Student Number

Hunters Hill High School

2020 Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions 11-16, show relevant mathematical reasoning and/or calculations

Total Marks: 100

Section I – 10 marks (pages 3-7)
Attempt all Questions 1–10
Allow about 15 minutes for this section

Section II – 90 marks (pages 8-13)
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Question 1–10.

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1. Imagine ω is an imaginary cube root of unity, then $(1+\omega-\omega^2)^{2020}$ is equal to

(A) $-2^{2020}\omega$

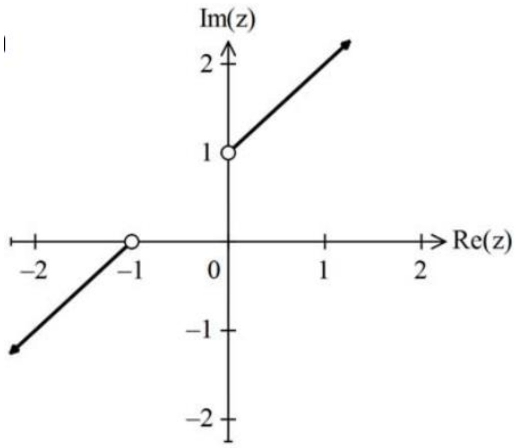
(B) $2^{2020}\omega$

(C) $-2^{2020}\omega^2$

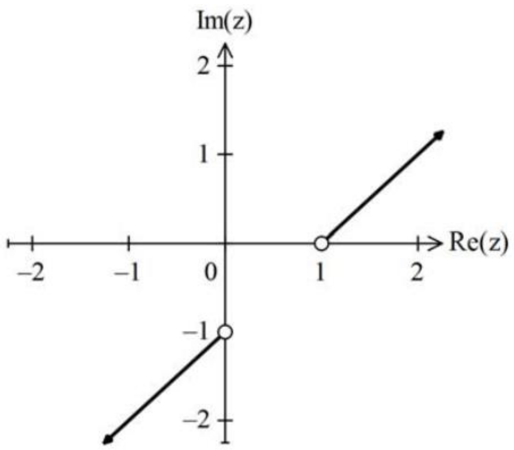
(D) $2^{2020}\omega^2$

2. Which of the following Argand diagrams describes the relationship defined by $\arg(z-i)=\arg(z+1)$?

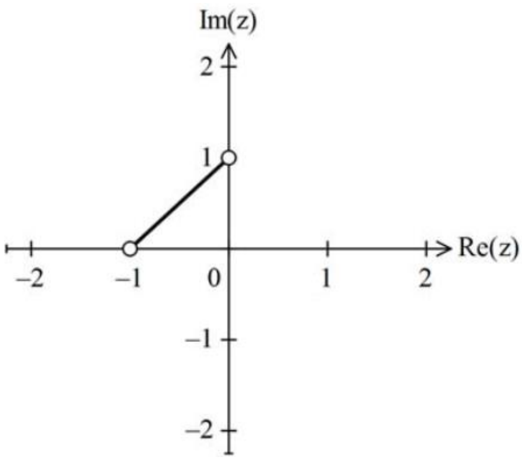
(A)



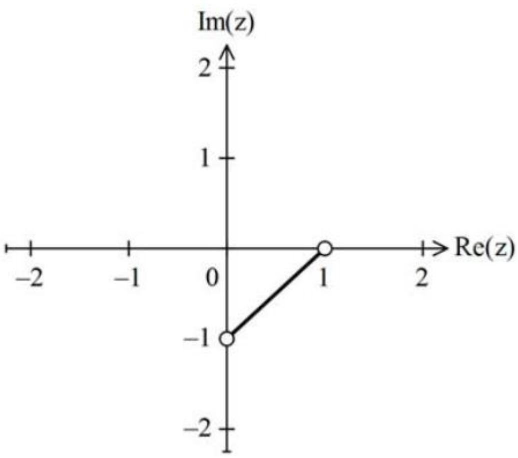
(B)



(C)



(D)



3. The acceleration of a particle moving in a straight line with velocity v is given by $\ddot{x} = \frac{1}{v}$. Which of the following functions best represents v in terms of x?

(A) $v = x + \dfrac{1}{3}$

(B) $v = \sqrt[3]{3x + 1}$

(C) $v = x + 1$

(D) $v = e^x$

4. Given $\underset{\sim}{a} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$, the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is:

(A) $\dfrac{1}{2}\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

(B) $\dfrac{7}{6}\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

(C) $\dfrac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

(D) $\dfrac{7}{6}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

5. If $a > b$ and $c > 0$, which of the following must be true?

I. $a^2 > b^2$

II. $a - c > b - c$

III. $\dfrac{a}{c^2} > \dfrac{b}{c^2}$

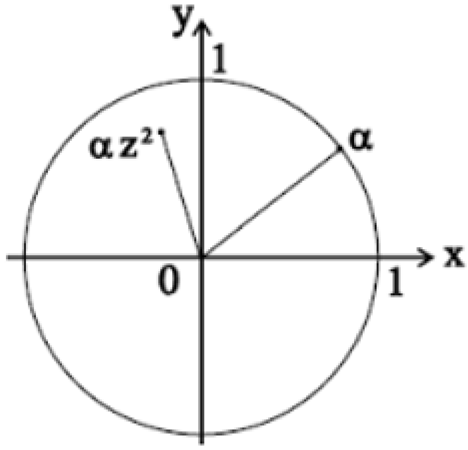
(A) I and II only

(B) II and III only

(C) I and III only

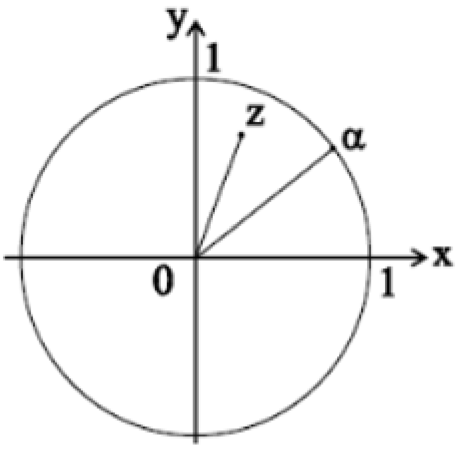
(D) I, II and III

6. The argand diagram below shows the complex numbers α and αz^2.

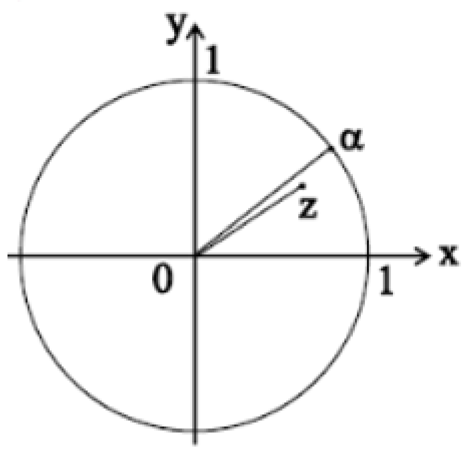


Which of the following best represents the positions of z and α?

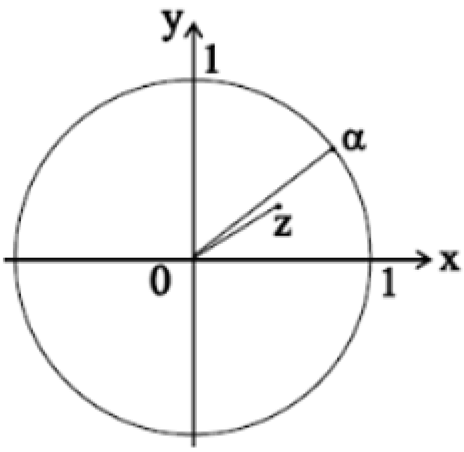
(A)



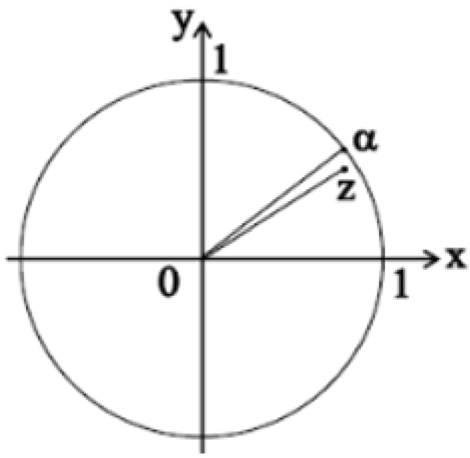
(B)



(C)



(D)



7. Which of the following expressions is equivalent to $\int_0^{2a} f(x)dx$?

(A) $\int_0^{2a} f(2a-x)dx$

(B) $\int_0^{2a} f(a-x)dx$

(C) $2\int_0^{2a} f(2a-x)dx$

(D) $2\int_0^{2a} f(a-x)dx$

8. What is the contrapositive of $\neg A \Rightarrow B$?

(A) $\neg B \Rightarrow \neg A$

(B) $B \Rightarrow \neg A$

(C) $\neg B \Rightarrow A$

(D) $B \Rightarrow A$

9. A sphere is defined by $\left|\underset{\sim}{r} - \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}\right| = 4\sqrt{3}$.

Which of the following points lie outside of the sphere?

(A) $(2, -3, 2)$

(B) $(5, -6, 4)$

(C) $(1, 2, 3)$

(D) $(1, 0, -3)$

10. A bob on a spring moves vertically in simple harmonic motion, with equation of motion given as $x = a \sin nt$. Which of the following best describes the initial placement and motion of the bob?

(A) the motion starts at the top and moves downwards

(B) the motion starts at the bottom and moves upwards

(C) the motion starts at the centre and moves upwards

(D) the motion starts at the centre and moves downwards

End of Section I

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a separate writing booklet.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Begin a new Writing Booklet

(a) Find $\int \frac{dx}{x^2 + 6x + 13}$. **2**

(b) Let $w = -\sqrt{3} + i$ and $z = 1 + i$

(i) Find wz in the form $a + ib$. **1**

(ii) Find w and z in mod-arg form. **2**

(iii) Hence, find the exact value of $\sin\frac{5\pi}{12}$. **2**

(c) Prove that the product of two consecutive even counting numbers is a multiple of 4. **2**

(d) **(i)** Find a, b and c such that $\frac{16}{(x^2 + 4)(2 - x)} = \frac{ax + b}{x^2 + 4} + \frac{c}{2 - x}$. **2**

(ii) Find $\int \frac{16}{(x^2 + 4)(2 - x)} dx$. **2**

(e) Solve $z^2 - 2iz + 2 = 0$. **2**

End of Question 11

Question 12 (15 marks) Begin a new Writing Booklet

(a) **(i)** Prove that $a^2 + b^2 \geq 2ab$. **1**

(ii) Hence, prove that $x^2 + y^2 + z^2 \geq xy + yz + zx$. **1**

(iii) Deduce that if $x + y + z = 1$, then $xy + yz + zx \leq \frac{1}{3}$. **2**

(b) Use integration by parts to show that $\int e^x \cos x \, dx = \frac{e^x}{2}(\sin x + \cos x)$. **3**

(c) Prove that $\sum_{k=1}^{n} r2^r = (n-1)(2^{n+1}) + 2$. **3**

(d) Sketch the region defined by the union of **3**

$$\frac{\pi}{6} \leq \arg z \leq \frac{5\pi}{6}$$
$$|z - i| \geq 2$$

(e) Suppose that $1 - i = e^{a+ib}$, where $a, b \in \mathbb{R}$ and $-\frac{\pi}{2} < b < \frac{\pi}{2}$. **2**
Find the exact values of a and b.

End of Question 12

Question 13 (15 marks) Begin a new Writing Booklet

(a) Find the angle between the vectors $\underset{\sim}{u} = (3, 5, -2)$ and $\underset{\sim}{v} = (-1, 2, 3)$. **2**

(b) **(i)** Find the square roots of $-24 - 10i$. **2**

(ii) Hence, or otherwise, solve $z^2 - (1 - i)z + 6 + 2i = 0$. **2**

(c) Consider the polynomial $P(z) = z^4 + 4z^3 + 14z^2 + 20z + 25$.

(i) It is known that $P(-1 + 2i) = 0$. **2**
Show that $P'(-1 + 2i) = 0$ and explain the significance of this.

(ii) Explain why $-1 - 2i$ is also a root of $P(z)$. **1**

(iii) Hence, factorise $P(z)$ over the complex numbers and then over the real. **2**

(d) **(i)** Find the coordinates of P that divides the interval AB in the ratio $2:3$, where $A = (1, 3, 4)$ and $B = (-4, 8, 2)$. **2**

(ii) Check the result by calculating $\left|\overrightarrow{AB}\right|$ and $\left|\overrightarrow{AP}\right|$. **2**

End of Question 13

Question 14 (15 marks) Begin a new Writing Booklet

(a) Prove by contradiction that $\log_3 4$ is irrational. **2**

(b) Use the substitution $u = e^x + 1$ to find $\int \frac{e^{2x}}{(e^x+1)^2}\,dx$. **3**

(c) **(i)** Show that for $k > 0$, **1**

$$\frac{1}{(k+1)^2} - \frac{1}{k} + \frac{1}{k+1} < 0$$

(ii) Use mathematical induction to prove that for all integers $n \geq 2$, **3**

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}.$$

(d) Using the substitution $t = \tan\frac{x}{2}$, or otherwise, find **3**

$$\int \frac{dx}{5 + 3\cos x}$$

(e) **(i)** Find the vector equation of $\overrightarrow{AB}$, given $A = (4, 8, 3)$ and $B = (5, 10, 4)$ in the form $\underset{\sim}{r_1} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \lambda \begin{bmatrix} d \\ e \\ f \end{bmatrix}$. **1**

(ii) Find the point of intersection of $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2} = \begin{bmatrix} 9 \\ 6 \\ 10 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$. **2**

End of Question 14

Question 15 (15 marks) Begin a new Writing Booklet

(a) Prove by the method of mathematical induction that for any positive integer $n > 0$, **3**

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

(b) A particle moving in simple harmonic motion has a period of π seconds. Initially the particle is at $x = 2$ with a velocity of 8 ms^{-1}.

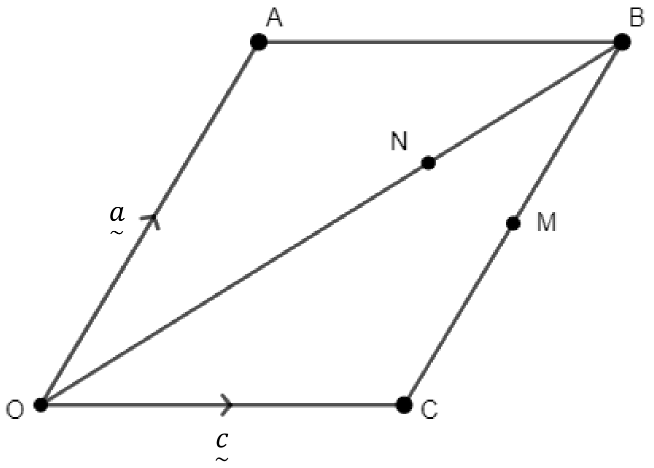
(i) Find x as a function of t in the form $x = b\cos nt + c\sin nt$. **2**

(ii) Find x as a function of t in the form $x = a\cos(nt - \alpha)$, where $a > 0$ and $0 \leq \alpha \leq 2\pi$. **2**

(iii) Hence, find the amplitude and the maximum speed of the particle. **2**

(c) $OABC$ is a parallelogram. $\overrightarrow{OA}$ is represented by the vector $\underset{\sim}{a}$ and $\overrightarrow{OC}$ is represented by the vector $\underset{\sim}{c}$.

M is the midpoint of BC and N is the point on OB such that $ON: NB = 2: 1$.



(i) Find expressions for the following vectors, given your answers in simplest form: $\overrightarrow{ON}$, $\overrightarrow{OM}$, $\overrightarrow{AN}$ and $\overrightarrow{AM}$. **4**

(ii) Show that the points A, N and M are collinear. **2**

End of Question 15

Question 16 (15 marks) Begin a new Writing Booklet

(a) A skydiver jumps out of a stationary balloon and starts falling freely to Earth.

She experiences gravity of mg and air resistance of $\dfrac{mv^2}{360}$ upwards.

Given that down is *positive*, $x = t = 0$ at the balloon and that $g = 9.8\ \text{ms}^{-2}$,

(i) Show that $\ddot{x} = g - \dfrac{v^2}{360}$ and find her terminal velocity. **2**

(ii) Show that $x = 180 \ln\left(\dfrac{360g}{360g - v^2}\right)$ and find the distance fallen when the skydiver reaches $50\ \text{ms}^{-1}$. **3**

(iii) Find the time taken for the skydiver to reach this speed. **3**

(b) Let $I_n = \displaystyle\int_0^{\frac{1}{2}} \frac{1}{(1+4x^2)^n}\,dx$, where n is a positive integer.

(i) Find the value of I_1. **2**

(ii) Using integration by parts, show that $I_n = \dfrac{2nI_{n+1}}{2n-1} + \dfrac{1}{2^{n+1}(1-2n)}$. **3**

$\left(\text{Hint: } \dfrac{m}{(1+m)^{n+1}} = \dfrac{1+m-1}{(1+m)^{n+1}}\right)$

(iii) Hence, evaluate $I_3 = \displaystyle\int_0^{\frac{1}{2}} \frac{1}{(1+4x^2)^3}\,dx$. **2**

End of Examination

MXX TRIAL – HHHS 2020 – SOLUTIONS

Q1. (D)

$$(1+\omega-\omega^2)^{2020}$$
$$=(1+\omega+\omega^2-2\omega^2)^{2020}$$
$$=(-2\omega^2)^{2020}$$
$$=2^{2020}\omega^{4040}$$
$$=2^{2020}\omega^2$$

$4040 \div 3 = 1346$ r2.

Q2. (A)

$\arg(z-i)=\arg(z+1)$

arg of z from i same as from -1.

Q3 (B)

$$\ddot{x}=\frac{1}{v}$$

$$\int v^2\,dv=\int dx$$

$$\frac{v^3}{3}=x+c$$

Q4 (A)

$$\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a}=\left(\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}\cdot\underset{\sim}{b}}\right)\underset{\sim}{b}$$

$$=\frac{1(3)-1(-2)+2(1)}{3^2+(-2)^2+1^2}\begin{bmatrix}3\\-2\\1\end{bmatrix}$$

$$=\frac{7}{14}\begin{bmatrix}3\\-2\\1\end{bmatrix}$$

Q5 (B)

$a > b \Rightarrow a^2 > b^2$ ✗ only if $a > b > 0$

$\Rightarrow a - c > b - c$ ✓

Q6 (D)

modulus must be closer to zero.

Q7 (A)

$$\int_0^{2a} f(2a-x)\,dx = -\left[F(2a-x)\right]_0^{2a}$$

$$= -F(2a-2a) + F(2a)$$

$$= F(2a) - F(0)$$

$$\int_0^{2a} f(x)\,dx = F(2a) - F(0)$$

Q8. (C)

$\neg B \Rightarrow \neg(\neg A)$

$\Rightarrow A.$

Q9. (D)

check distance from $(2, -3, 5)$

$d^2 > 48$

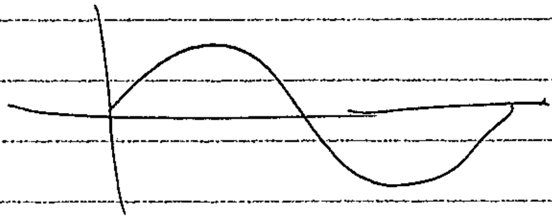
A: $d^2 = 0^2 + 0^2 + 3^2 = 9$

B: $d^2 = 3^2 + (3)^2 + 1^2 = 19$

C: $d^2 = 1^2 + (-5)^2 + 2^2 = 30$

D: $d^2 = 1^2 + (3)^2 + 8^2 = 74$.

Q10. (C.)



Q11 a)

$$\int \frac{dx}{x^2+6x+13} = \int \frac{dx}{x^2+6x+9+4}$$

$$= \int \frac{dx}{(x+3)^2+4}$$

$$= \frac{1}{2}\tan^{-1}\frac{x+3}{2} + c$$

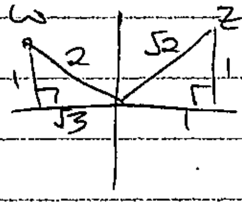
b.) i) $w = -\sqrt{3}+i,\ z = 1+i$

$$wz = (-\sqrt{3}+i)(1+i)$$

$$= -\sqrt{3} + i - \sqrt{3}i - 1$$

$$= -(1+\sqrt{3}) + i(1-\sqrt{3}) \quad \text{— i}$$

ii)



$$w = 2\,\text{cis}\,\frac{5\pi}{6}$$

$$z = \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$$

iii) $$wz = 2\sqrt{2}\,\text{cis}\left(\frac{5\pi}{6} + \frac{\pi}{4}\right)$$

$$= 2\sqrt{2}\,\text{cis}\left(\frac{10\pi + 3\pi}{12}\right)$$

$$= 2\sqrt{2}\,\text{cis}\,\frac{13\pi}{12}. \quad \text{— ii}$$

equating Im components of i and ii

$2\sqrt{2}\sin\frac{13\pi}{12} = (1-\sqrt{3})$

$\therefore \sin\frac{13\pi}{12} = \frac{1-\sqrt{3}}{2\sqrt{2}}$

c) let two consecutive even numbers be

$2n,\ 2n+2$ for $n \in \mathbb{Z}$

product $= 2n(2n+2)$

$= 4n^2 + 4n$

$= 4(n^2+n)$

hence the product is a multiple of 4

d) i) let $\frac{16}{(x^2+4)(2-x)} = \frac{ax+b}{x^2+4} + \frac{c}{2-x}$

$\therefore 16 = (ax+b)(2-x) + c(x^2+4)$

$= 2ax + 2b - ax^2 - xb + cx^2 + 4c.$

for $x=2$.

$16 = 8c$

$c = 2.$

constants: $16 = 2b + 4c \implies 2b = 8$

$b = 4$

x-term: $0 = 2a - b \rightarrow 2a = 4$

$a = 2$

$\therefore a=2, b=4, c=2.$

ii) $\int \frac{16}{(x^2+4)(2-x)}\,dx = \int \left(\frac{2x+4}{x^2+4} + \frac{2}{2-x}\right) dx$

$= \ln|x^2+4| + 2\tan^{-1}\frac{x}{2} - 2\ln|2-x| + c$

e) $z^2 - 2iz + 2 = 0$

$z = \frac{2i \pm \sqrt{(2i)^2 - 4(1)(2)}}{2(1)}$

$= \frac{2i \pm \sqrt{-12}}{2}$

$= \frac{2i \pm 2\sqrt{3}i}{2}$

$= (1+\sqrt{3})i,\ (1-\sqrt{3})i$

alternatively,

$z^2 - 2iz - 1 = -2 - 1$

$(z-i)^2 = -3$

$z - i = \pm\sqrt{3}i$

$z = (1 \pm \sqrt{3})i$

Q12 a i) $\forall a, b \in \mathbb{R}$

$(a-b)^2 \geq 0$

$a^2 - 2ab + b^2 \geq 0$

$\therefore a^2 + b^2 \geq 2ab$

ii) from (i)

$x^2 + y^2 \geq 2xy$ — i

$y^2 + z^2 \geq 2yz$ — ii

$z^2 + x^2 \geq 2zx$ — iii

i+ii+iii

$2x^2 + 2y^2 + 2z^2 \geq 2xy + 2yz + 2zx$

$\therefore x^2 + y^2 + z^2 \geq xy + yz + zx$

iii $$(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx)$$

$$\therefore\ x^2+y^2+z^2 = 1-2(xy+yz+zx)$$

from (ii)

$$1-2(xy+yz+zx) \geqslant xy+yz+zx.$$

$$\therefore\ 3(xy+yz+zx) \leqslant 1$$

$$xy+yz+zx \leqslant \frac{1}{3}$$

b). $\int e^x \cos x\,dx$ let $u=e^x$ $dv=\cos x\,dx$
$du=e^x dx$ $v=\sin x.$

$$=e^x\sin x - \int e^x \sin x\,dx$$

let $dv = \sin x\,dx$
$v=-\cos x$

$$=e^x\sin x - \left(-e^x\cos x - \int e^x(-\cos x)dx\right)$$

$$=e^x\sin x + e^x\cos x - \int e^x\cos x\,dx$$

$$\therefore\ 2\int e^x\cos x\,dx = e^x(\sin x+\cos x)$$

$$\int e^x\cos x\,dx = \frac{e^x}{2}(\sin x+\cos x)$$

c) Prove. $\sum_{k=1}^{n} k2^k = (n-1)(2^{n+1})+2$.

Prove true for . n=1

LHS = 1×2^1 RHS = $(1-1)2^{1+1}+2$

$=2$ $=2$

$=$ LHS

$\therefore$ true for n=1

Assume true for n=k

i.e. $\sum_{k=1}^{k} k2^k = (k-1)(2^{k+1})+2$.

Prove true for n=k+1

i.e. $\sum_{r=1}^{k+1} (k+1)2^{k+1} = (k+1-1)(2^{(k+1)+1})+2$

$= k2^{k+2}+2$

LHS = $(k+1)2^{k+1} + \sum_{r=1}^{k} r2^r$

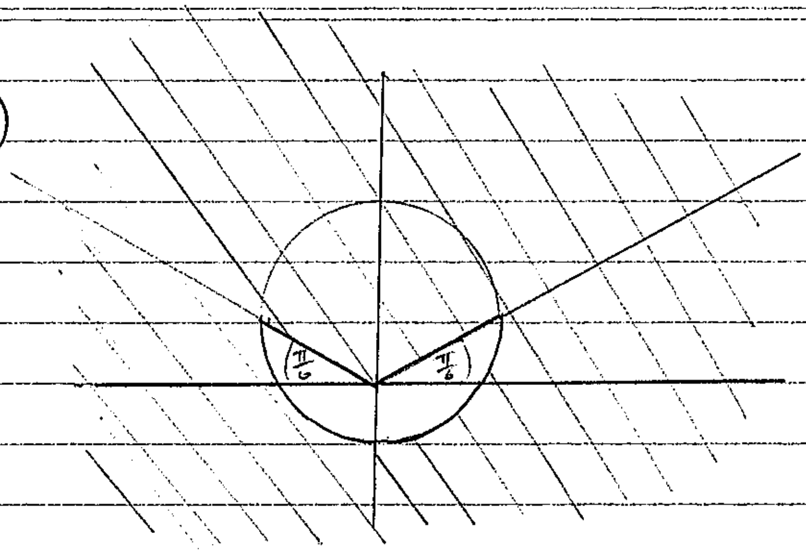
$= (k+1)2^{k+1} + (k-1)(2^{k+1})+2$ by induction hypothesis

$= (k+1+(k-1)).2^{k+1}+2$

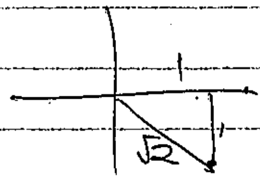
$= 2k.2^{k+1}+2$

$= k2^{k+2}+2$ = RHS

$\therefore$ by principle of induction, this is true d)



e.) $1 - i = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$



$= \sqrt{2}\, e^{-\frac{\pi}{4}i}$

$= e^{\ln\sqrt{2}} \cdot e^{-\frac{\pi}{4}i}$

$= e^{\ln\sqrt{2} - \frac{\pi}{4}i}$

$\therefore\ a = \ln\sqrt{2}$
$b = -\frac{\pi}{4}$

Q 13 a) $3(-1) + (5)(2) - 2(3) = \sqrt{3^2+5^2+(-2)^2}\,\sqrt{(-1)^2+2^2+3^2}\,\cos\theta.$

$1 = \sqrt{38}\ \sqrt{14}\ \cos\theta.$

$\cos\theta = \frac{1}{\sqrt{532}}$

$\theta = 1.527$ radians
$= 87°30'54.48''$

b) i) let $(x+iy)^2 = -24-10i$

$$x^2-y^2+2xyi = -24-10i$$

equate Re + Im

$$x^2-y^2 = -24 \quad \text{—i}$$

$$2xy = -10 \quad \Rightarrow x^2y^2 = 25 \quad \text{—ii}$$

from i

$$x^4 - x^2y^2 = -24x^2$$

$$x^4 - 25 = -24x^2$$

$$x^4 + 24x^2 - 25 = 0$$

$$(x^2+25)(x^2-1) = 0$$

$$x^2 = 1 \qquad (x^2 \neq -25)$$

$$x = \pm 1$$

$$\Rightarrow y = \mp 5$$

$$\therefore \sqrt{-24-10i} = \pm(1 \mp 5i)$$

ii) $z^2 - (1-i)z + 6+2i = 0$

$$z = \frac{(1-i) \pm \sqrt{(1-i)^2 - 4(1)(6+2i)}}{2(1)}$$

$$= \frac{1-i \pm \sqrt{-2i - 24 - 8i}}{2}$$

$$= \frac{1-i \pm \sqrt{-24-10i}}{2}$$

$$= \frac{1-i \pm (1-5i)}{2}$$

$$= \frac{2-6i}{2}, \frac{4i}{2}$$

$$= 1-3i, 2i$$

c) $P(z) = z^4 + 4z^3 + 14z^2 + 20z + 25$.

i) $P(-1+2i) = 0$

$$P'(z) = 4z^3 + 12z^2 + 28z + 20$$

$$P'(-1+2i) = 4(-1+2i)^3 + 12(-1+2i)^2 + 28(-1+2i) + 20$$

$$= 4(-1 + 3(2i) - 3(-4) - 8i) + 12(1 - 4i - 4) + 28(-1+2i) + 20$$

$$= 44 - 8i + (-36) - 48i - 28 + 56i + 20$$

$$= 0$$

this means that $-1+2i$ is a double root.

ii) $-1-2i$ is also a root as

I it is the conjugate of $-1+2i$

II the polynomial has real coefficients

and complex roots of polynomials with real coefficients occur in conjugate pairs

iii) $\therefore P(z) = \left(z-(-1-2i)\right)^2\left(z-(-1+2i)\right)^2$

$= (z+1+2i)^2(z+1-2i)^2$ over complex

$= \left((z+1)^2-(2i)^2\right)^2$

$= \left(z^2+2z+1-(-4)\right)^2$

$= (z^2+2z+5)^2$ over real

d) i) $A=(1,3,4)$ $\quad$ $AB \Rightarrow 2:3$

$B=(-4,8,2)$

$P = \frac{1}{2+3}\left(3(1,3,4)+2(-4,8,2)\right)$

$= \frac{1}{5}\left((3,9,12)+(-8,16,4)\right)$

$= \frac{1}{5}(-5,25,16)$

$= (-1,5,3\tfrac{1}{5})$

ii) $\vec{AB} = \begin{bmatrix}-4\\8\\2\end{bmatrix} - \begin{bmatrix}1\\3\\4\end{bmatrix}$

$= \begin{bmatrix}-5\\5\\-2\end{bmatrix}$

$|\vec{AB}| = \sqrt{(-5)^2+5^2+(-2)^2}$

$= \sqrt{54}$

$\vec{AP} = \begin{bmatrix}-1\\5\\\frac{16}{5}\end{bmatrix} - \begin{bmatrix}1\\3\\4\end{bmatrix}$

$= \begin{bmatrix}-2\\2\\-\frac{4}{5}\end{bmatrix}$

$|\vec{AP}| = \sqrt{(-2)^2+2^2+\left(-\frac{4}{5}\right)^2}$

$= \sqrt{8+\frac{16}{25}}$

$$|\overrightarrow{AP}| = \sqrt{\frac{216}{25}}$$

$$= \frac{2}{5}\sqrt{54}$$

$$= \frac{2}{5}|\overrightarrow{AB}|$$

Q14.

a). let $\log_3 4 = \frac{a}{b}$, $a, b \in \mathbb{Z}$

$$b \log_3 4 = a$$

$$\log_3 4^b = a$$

$$4^b = 3^a$$

$$2^{2b} = 3^a$$

given the prime nature of 2 and 3, and $a, b \in \mathbb{Z}$, this cannot be true

$\therefore$ $\log_3 4$ is irrational

b) $I = \int \frac{e^{2x}}{(e^x+1)^2} dx$ let $u = e^x+1$, $e^x = u-1$
$du = e^x dx$

$$= \int \frac{e^x}{(e^x+1)^2} \cdot e^x dx$$

$$= \int \frac{u-1}{u^2} du$$

$$= \int \left(\frac{1}{u} - \frac{1}{u^2}\right) du$$

$= \ln|u| + \frac{1}{u} + c$

$= \ln(e^x+1) + \frac{1}{e^x+1} + c$

c) i. Show $\frac{1}{(k+1)^2} - \frac{1}{k} + \frac{1}{k+1} < 0$

$LHS = \frac{1}{(k+1)^2} - \frac{1}{k} + \frac{1}{k+1}$

$= \frac{k - (k+1)^2 + k((k+1))}{k(k+1)^2}$

$= \frac{k - k^2 - 2k - 1 + k^2 + k}{k((k+1)^2}$

$= \frac{-1}{k((k+1)^2} < 0$ as $k > 0$

ii) Prove $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} < 2 - \frac{1}{n}$, $n \geq 2$

Prove true for $n=2$

$LHS = \frac{1}{1^2} + \frac{1}{2^2}$ $\quad$ $RHS = 2 - \frac{1}{n}$

$= \frac{5}{4}$ $\quad$ $= \frac{3}{2}$

as $\frac{5}{4} < \frac{3}{2}$, then statement true for $n=2$

Assume true for $n=k$

$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{k^2} < 2 - \frac{1}{k}$ — induction hypothesis

Prove true for $n=k+1$

i.e. $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{k^2}+\frac{1}{(k+1)^2}<2-\frac{1}{k+1}$

$LHS<2-\frac{1}{k}+\frac{1}{(k+1)^2}$, using induction hypothesis

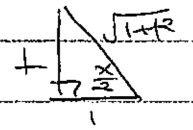
$=2+\frac{1}{(k+1)^2}-\frac{1}{k}+\frac{1}{k+1}-\frac{1}{k+1}$

$<2-\frac{1}{k+1}$ from (i)

$<RHS$

$\therefore$ by principle of induction, statement is true for $n\geq 2$

d) $I=\int\frac{dx}{5+3\cos x}$ let $t=\tan\frac{x}{2}$



$\therefore \cos x=\frac{1-t^2}{1+t^2}$ $dt=\frac{1}{2}\sec^2\frac{x}{2}\,dx$

$dx=\frac{2dt}{1+t^2}$

$I=\int\frac{1}{5+\frac{3(1-t^2)}{1+t^2}}\cdot\frac{2dt}{1+t^2}$

$=2\int\frac{(1+t^2)\,dt}{(5(1+t^2)+3(1-t^2))(1+t^2)}$

$=2\int\frac{dt}{8+2t^2}$

$=\int\frac{dt}{4+t^2}$

$= \frac{1}{2} \tan^{-1} \frac{t}{2} + c.$

$= \frac{1}{2} \tan^{-1} \left[\frac{\tan \frac{x}{2}}{2} \right] + c$

e) i). $A = (4,8,3)$, $B = (5,10,4)$

$\underset{\sim}{r_1} = A + \lambda \overrightarrow{AB}$

$= \begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

ii) equate $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$

$\begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 10 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}.$

$\therefore \; 4 + \lambda = 9 + \mu$ — i

$8 + 2\lambda = 6 - 4\mu$ — ii

$3 + \lambda = 10 + 2\mu$ — iii

i − iii

$1 = -1 - \mu$

$\mu = -2. \Rightarrow \lambda = 3$ (from i)

test in ii $LHS = 8 + 2(3)$ $RHS = 6 - 4(-2)$

$= 14$ $= 14.$

iii $LHS = 3 + 3$ $RHS = 10 + 2(-2)$

$= 6$ $= 6$

$\therefore$ intersection at $\begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \\ 6 \end{bmatrix}$

Q15.a) Prove $\frac{d}{dx}(x^n) = nx^{n-1}$.

Prove true for $n=$

$LHS = \frac{d}{dx}(x^1)$ $\quad RHS = 1 \cdot x^0$

$= 1$ $\quad = 1$

$= LHS$

$\therefore$ true for $n=1$

Assume true for $n=k$

$\therefore \frac{d}{dx}(x^k) = kx^{k-1}$

Prove true for $n=k+1$

i.e. $\frac{d}{dx}(x^{k+1}) = (k+1)x^{(k+1)-1}$

$LHS = \frac{d}{dx}(x^{k+1})$

$= \frac{d}{dx}(x \cdot x^k)$

$= x^k \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(x^k)$

$= x^k \cdot 1 + x \cdot kx^{k-1}$ using induction hypothesis

$= x^k + kx^k$

$= (k+1)x^k$

$= (k+1)x^{(k+1)-1} = RHS$

$\therefore$ by induction, true for $n \geq 1$

b) i) $x = b\cos nt + c\sin nt$

$t=0, x=2$

$2 = b\cos 0 + c\sin 0$

$= b.$

$v = -bn\sin nt + cn\cos nt$

$t=0, v=8$

$8 = -bn\sin 0 + cn\cos 0$

$\therefore 8 = cn$

as period is π, $n=2$

$\therefore b=2, c=4$

and $x = 2\cos 2t + 4\sin 2t$

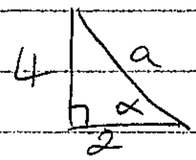
ii) $x = a\cos(nt-\alpha)$

$= a\cos nt.\cos\alpha + a\sin nt.\sin\alpha.$

equating coefficients

$a\cos\alpha = 2$

$a\sin\alpha = 4$



$\therefore a = 2\sqrt{5}$

$\tan\alpha = 2 \Rightarrow \alpha = \tan^{-1}2, \; 0<\alpha<\frac{\pi}{2}$

as $\cos\alpha > 0$, $\sin\alpha > 0$

$\therefore x = 2\sqrt{5}\cos(2t - \tan^{-1}2)$

iii. for SHM, max speed when $x=0$.

$$\text{so } 2\sqrt{5}\cos(2t-\tan^{-1}2)=0$$
$$2t-\tan^{-1}2=1$$
$$t=\frac{1}{2}(1+\tan^{-1}2)$$

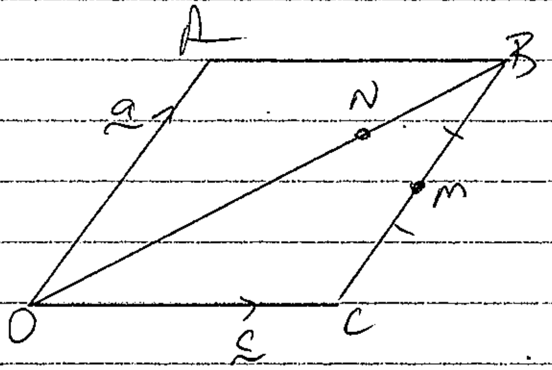
$$v=-4\sqrt{5}\sin(2t-\tan^{-1}2)$$

$$=-4\sqrt{5}\sin\left(2\cdot\frac{1}{2}(1+\tan^{-1}2)-\tan^{-1}2\right)$$

$$=-4\sqrt{5}\sin 1$$
$$=-7.5263453$$

$\therefore$ amplitude is $2\sqrt{5}$ units
and max speed is $7.53\ ms^{-1}$ (2dp)

c) i.



$$\overrightarrow{ON}=\frac{2}{3}(\underset{\sim}{a}+\underset{\sim}{c})$$

$$\overrightarrow{OM}=\frac{1}{2}\underset{\sim}{a}+\underset{\sim}{c}$$

$$\overrightarrow{AN}=\overrightarrow{ON}-\overrightarrow{OA}$$
$$=\frac{2}{3}(\underset{\sim}{a}+\underset{\sim}{c})-\underset{\sim}{a}$$
$$=-\frac{1}{3}\underset{\sim}{a}+\frac{2}{3}\underset{\sim}{c}$$

$$\overrightarrow{AM}=\overrightarrow{OM}-\overrightarrow{OA}$$
$$=\frac{1}{2}\underset{\sim}{a}+\underset{\sim}{c}-\underset{\sim}{a}$$
$$=-\frac{1}{2}\underset{\sim}{a}+\underset{\sim}{c}$$

iii) $\overrightarrow{AN} = -\frac{1}{3}\underset{\sim}{a} + \frac{2}{3}\underset{\sim}{c}$

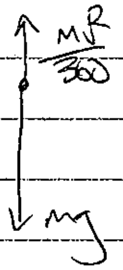
$= \frac{2}{3}\left(-\frac{1}{2}\underset{\sim}{a} + \underset{\sim}{c}\right)$

$= \frac{2}{3}\overrightarrow{AM}$

as both vectors originate at A.

$\therefore$ A, N, and M are collinear.

Q16 a) i)



$F = ma$

$ma = mg - \frac{mv^2}{360}$

$a = g - \frac{v^2}{360}$

for terminal velocity, $a = 0$

$\therefore g = \frac{v^2}{360}$

$v^2 = 360g$

$v = \sqrt{360g}$

$= 59.39696 \text{ ms}^{-1}$

ii) $v\frac{dv}{dx} = g - \frac{v^2}{360}$

$= \frac{360g - v^2}{360}$

$\int \frac{v\,dv}{360g - v^2} = \int \frac{dx}{360}$

$$-\frac{1}{2}\ln\left|360g - v^2\right| = \frac{x}{360} + c$$

given $t=0 \Rightarrow x=0, v=0$

then $c = -\frac{1}{2}\ln 360g$

$$\frac{x}{360} = \frac{1}{2}\ln 360 - \frac{1}{2}\ln(360g - v^2)$$

$$= \frac{1}{2}\ln\frac{360g}{360g - v^2}$$

$$\therefore x = 180\ln\frac{360g}{360g - v^2}$$

at $v = 50$

$$x = 180\ln\frac{360g}{360g - 50^2}$$

$$= 221.96087448\text{ m} \quad \text{below the balloon}$$

ii)

$$\ddot{x} = \frac{360g - v^2}{360}$$

$$\therefore \frac{dv}{dt} = \frac{360g - v^2}{360}$$

$$dt = \frac{360\,dv}{360g - v^2}$$

$$t = \int\frac{360\,dv}{(360g - v^2}$$

let $\frac{360}{360g - v^2} = \frac{A}{\sqrt{360g} - v} + \frac{B}{\sqrt{360g} + v}$

$$\therefore 360 = A(\sqrt{360g} + v) + B(\sqrt{360g} - v)$$

when $v = \sqrt{360g}$ $\quad A = \dfrac{360}{2\sqrt{360g}}$

$= \dfrac{3\sqrt{10}}{\sqrt{g}}$

when $v = -\sqrt{360g}$ $\quad B = \dfrac{3\sqrt{10}}{\sqrt{g}}$

$$\therefore t = \int \left(\frac{3\sqrt{10}}{\sqrt{g}(\sqrt{360g} - v)} + \frac{3\sqrt{10}}{\sqrt{g}(\sqrt{360g} + v)} \right) dx$$

$$= \frac{3\sqrt{10}}{\sqrt{g}} \left(-\ln(\sqrt{360g} - v) + \ln(\sqrt{360g} + v) \right) + c$$

$$= \frac{3\sqrt{10}}{\sqrt{g}} \ln \frac{\sqrt{360g} + v}{\sqrt{360g} - v} + c.$$

at $t=0, v=0$ $\Rightarrow$ $c = 0$ $\quad (\ln 1 = 0)$

at $v = 50$

$$t = \frac{3\sqrt{10}}{\sqrt{g}} \ln \frac{\sqrt{360g} + 50}{\sqrt{360g} - 50}$$

$= 7.438548983$

$\therefore$ skydiver reaches $50\,ms^{-1}$ after $7.44\,s$ (2dp)

(b) i. $I_n = \int_0^{\frac{1}{2}} \frac{1}{(1+4x^2)^n} \, dx$

$$I_1 = \int_0^{\frac{1}{2}} \frac{1}{1+4x^2} \, dx$$

$$= \frac{1}{2}\left[\tan^{-1} 2x\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{2}\left(\tan^{-1} 2(\tfrac{1}{2}) - \tan^{-1} 2(0)\right)$$

$$= \frac{1}{2}.\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\pi}{8}$$

ii. let $u = (1+4x^2)^{-n}$ $\qquad dv = dx$

$$du = -n(8x)(1+4x^2)^{-n-1} \, dx \qquad v = x.$$

$$= \frac{-8nx}{(1+4x^2)^{n+1}}$$

$$I_n = \left[\frac{x}{(1+4x^2)^n}\right]_0^{\frac{1}{2}} + 2n\int_0^{\frac{1}{2}} \frac{4x^2}{(1+4x^2)^{n+1}} \, dx.$$

$$= \frac{\frac{1}{2}}{(1+4(\frac{1}{2})^2)^n} - \frac{0}{(1+4(0))^n} + 2n\int_0^{\frac{1}{2}} \frac{1+4x^2-1}{(1+4x^2)^{n+1}} \, dx$$

$$= \frac{1}{2\times 2^n} + 2n\int_0^{\frac{1}{2}} \frac{1+4x^2}{(1+4x^2)^{n+1}} \, dx - 2n\int_0^{\frac{1}{2}} \frac{1}{(1+4x^2)^{n+1}} \, dx$$

$$= \frac{1}{2^{n+1}} + 2n\int_0^{\frac{1}{2}} \frac{dx}{(1+4x^2)^n} - 2n\int_0^{\frac{1}{2}} \frac{dx}{(1+4x^2)^{n+1}}$$

$$\therefore I_n = \frac{1}{2^{n+1}} + 2n\, I_n - 2n\, I_{n+1}$$

$$(2n-1)I_n = 2n\, I_{n+1} - \frac{1}{2^{n+1}}$$

$$\therefore I_n = \frac{2n\, I_{n+1}}{2n-1} - \frac{1}{2^{n+1}(2n-1)}$$

$$= \frac{2n\, I_{n+1}}{2n-1} + \frac{1}{2^{n+1}(1-2n)}$$

iii) $$I_1 = \frac{2(1)\, I_2}{2(1)-1} + \frac{1}{2^{1+1}(1-2(1))}$$

$$\frac{\pi}{8} = 2I_2 - \frac{1}{4}$$

$$I_2 = \frac{1}{2}\left(\frac{\pi}{8} + \frac{1}{4}\right)$$

$$= \frac{\pi+2}{16}$$

$$I_2 = \frac{2(2)\, I_3}{2(2)-1} + \frac{1}{2^{2+1}(1-2(2))}$$

$$\frac{\pi+2}{16} = \frac{4}{3}I_3 - \frac{1}{3\times 2^3}$$

$$\frac{4}{3}I_3 = \frac{\pi+2}{16} + \frac{1}{24}$$

$$= \frac{3\pi+6+2}{48}$$

$$I_3 = \frac{3}{4}\frac{(3\pi+8)}{48}$$

$$= \frac{3\pi+8}{64}$$